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THE DIFFRACTION PROBLEM FOR A SUBMERGED CIRCULAR CYLINDER SOLVED BY SPLINE COLLOCATION

by

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1. Introduction

In 1948 Dean /1/ discovered that a submerged, restrained circular cylinder does not lead to any reflection of linear surface waves. This result has been confirmed by later works, where three different solution methods have been used.

Dean/1/ and Mehlum /2/ transforms the fluid domain into a region bounded by two circles and then solves the problem in this region.

Ursell /3/ and Ogilvie /4/ use the multipole method, i.e. the potential is expanded in a modified Laurent series around the centre of the cylinder.

Schnute /5/ and Levine /6/ use the integral equation method with a modified source potential as the Greens function. They solve the integral equation by expanding the potential in a Fourier series.

When we started our studies on this problem we wanted to use a solution method which can be extended in two directions. It should be possible to handle submerged cylinders of different shapes, and it should be possible to study the full second-order problem without too many numerical difficulties.
The first extension suggest that the integral equation method should be used and that the equation should be solved by polynomial collocation. The easiest choice is then to approximate the potential by a piecewise constant function. This method has been extensively used in the solution of the radiation problem for a semi-submerged cylinder (see f.ex. Potash /7/). This is by far the easiest method to extend to different shapes because the contour can be replaced by straight lines. But in Vada /9/ a discussion is given which concludes that this method is not easily extended to second order. This is due to the fact that both the first and second derivatives of the forst-order potential is needed to find the second-order potential. Thereby accuracy is lost and numerical approximations to the derivatives are needed. We therefore want to use a more accurate method. The natural choice then is to approximate the potential with a cubic spline. In addition to the better accuracy this also gives a solution with continuous first and second derivatives. It is also possible to extend the method to other types of contours without too much difficulty, although in some cases this is not as easy as with the piecewise constant.

In this report, which is the first step on the road towards the more difficult problems, we have only studied linear waves passing over a restrained circular cylinder. The resulting phase-shift and first-order force have been computed several times with different methods and they therefore gives a useful check of our method. The good results which have been achieved have encouraged us to pass on to the more difficult problems.

Parallell to our work Grue & Palm /8/ has studied the same problem, but with a vortex distribution instead of our source distribution. This results in an ordinary integral equation whereas we have to deal with a singular one.

2. Formulation of the problem

The problem we want to study is the following:

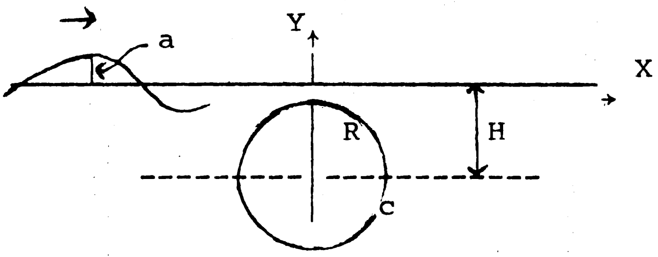


Fig. 2-1

A linear wave with amplitude a and frequency ω is propagating over a totally submerged, fixed circular cylinder with radius R and its centre at $Y = -H$. The fluid is ideal, the motion is irrotational and all boundary conditions are linearized.

We introduce the dimensionless variables

$$(2\text{-}1) \qquad x = \frac{X}{R}\,, \quad y = \frac{Y}{R}\,, \quad \tau = \omega t\,, \quad \varepsilon = \frac{a}{R}\,, \quad K = \frac{R\omega^2}{g}$$

and the dimensionless potentials

$$(2\text{-}2) \qquad \hat{\Phi}_i(x,y,\tau) = \frac{1}{\omega R^2}\,\Phi_i(X,Y,t) \qquad i = 0,7$$

The time-dependence is separated from the equations by writing

$$(2\text{-}3) \qquad \hat{\Phi}_i = \mathrm{Re}\{\varepsilon\phi_i(x,y)e^{-jz}\} \qquad i = 0,7$$

The potential of the incoming wave is now

$$(2\text{-}4) \qquad \phi_0(x,y) = -\frac{1}{K}\,e^{ky}\,e^{jK(x-x_0)}$$

and the total (time-independent) velocity potential is

$$\phi(x,y) = \phi_0(x,y)+\phi_7(x,y)$$

where the diffraction potential, ϕ_7 , is the solution of the problem

$$\nabla^2\phi_7(x,y) = 0 \qquad \text{in the fluid}$$

(2-5) $$\frac{\partial\phi_7}{\partial n} = -\frac{\partial\phi_0}{\partial n} \qquad \text{on } C$$

$$\frac{\partial\phi_7}{\partial y} - K\phi_7 = 0\,, \qquad y = 0$$

$$\frac{\partial\phi_7}{\partial n} = 0\,, \qquad y = -\infty$$

$$\frac{\partial\phi_7}{\partial n} \mp iK\phi_7 = 0\,, \qquad x = \pm\infty$$

(see f.ex. Ogilvie /4/ or Newman /11/).

This problem is solved by the introduction of the Greens function

$$\sigma(z,\zeta,K) = \mathrm{Re}\left\{\ln(z-\zeta)-\ln(z-\bar{\zeta})+2\int_0^\infty \frac{e^{-i\nu(z-\bar{\zeta})}}{\kappa-\nu}\,d\nu-2\pi j e^{-iK(z-\zeta)}\right\}$$

(see Wehausen & Laitone /10/). From Greens theorem we then obtain the following integral equation for ϕ_7 :

(2-6) $$-\pi\,\phi_7(z)+\int_C\phi_7(\zeta)\frac{\partial\sigma}{\partial n(\zeta)}(z,\zeta,K)ds(\zeta) = \int_C\frac{\partial\phi_7}{\partial n(\zeta)}\,\sigma(z,\zeta,K)ds(\zeta)$$

which is a Fredholm equation of the second kind. We now want to solve this equation.

3. The solution method

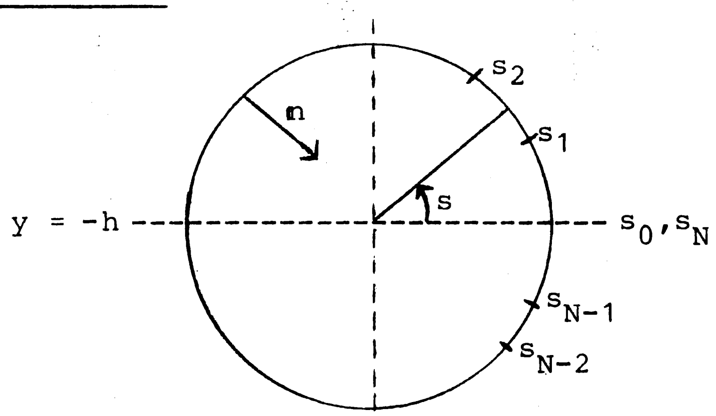


Fig. 3-1

We now divide the cylinder contour into N elements and approximates ϕ_7 by the spline

(3-1) $$\phi_7(s) \approx \sum_{i=1}^{N} q_i B_i(s)$$

where $B_i(s)$ is the cubic B-spline with its maximum at s_i (see appendix). At the nodal points we then have

$$\phi_7(s_i) = \frac{1}{6} q_{i-1} + \frac{4}{6} q_i + \frac{1}{6} q_{i+1} \ .$$

We now require the integral equation (2-6) to be satisfied at all nodal points. This gives the set of equations

$$-\pi\left(\frac{1}{6} q_{i-1} + \frac{4}{6} q_i + \frac{1}{6} q_{i+1}\right)$$

$$+ \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} \left(\sum_{k=1}^{N} q_k B_k\right) \frac{\partial G}{\partial n}(z_i, \zeta, K) ds(\zeta)$$

$$= \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} \frac{\partial \phi_7}{\partial n} G(z_i, \zeta, K) ds$$

In the sum over K only B_{j-2}, B_{j-1}, B_j and B_{j+1} contributes to the integral, since all the others are zero in the interval $[s_{j-1}, s_j]$. We therefore have to solve the equations

$$-\frac{\pi}{6}(q_{i-1} + 4q_i + q_{i+1}) + A_{ij} q_j = B_{ij} \qquad i,j = 1,2,\ldots,N$$

where

$$A_{ij} = \sum_{k=j-2}^{j+1} q_k \int_{s_{j-1}}^{s_j} B_k \frac{\partial G}{\partial n}(z_i, \zeta, K) ds(\zeta)$$

$$B_{ij} = \int_{s_{j-1}}^{s_j} \frac{\partial \phi_7}{\partial n} G(z_i, \zeta_K) ds(\zeta)$$

In the k-sum in A_{ij} we use the periodicity of ϕ_7 which gives

$$q_{i+N} = q_i$$

The integrals are computed by the 3-point Gauss formula, a method which is both computationally efficient and sufficiently accurate. The value of $\frac{\partial \phi_7}{\partial n}$ are found from the boundary condition (2-5).

Finally the equations are solved by ordinary elimination, and when the q_i-s are known the value of the potential can be computed anywhere on the contour from (1).

4. Computation of the forces

We make the pressure dimensionless by setting

$$\hat{p}(x,y,\tau) = \frac{p}{\rho g R}$$

which together with (2-1) and (2-2) gives the dimensionless Bernoulli equation

$$\hat{p} = -K\left(\frac{\partial\hat{\Phi}}{\partial\tau} + \tfrac{1}{2}(\nabla\hat{\Phi})^2\right) \tag{4-1}$$

where $\hat{\Phi}(x,y,\tau) = \hat{\Phi}_0+\hat{\Phi}_7$

The hydrostatic term is neglected here because we assume that the hydrostatic force is compensated by the weight of the cylinder. The dimensionless force per unit length of the cylinder is

$$\vec{F}(\tau) = \frac{\vec{F}(t)}{\rho g\ R^2} = \varepsilon\ \vec{F}_1+\varepsilon^2\ \vec{F}_2+O(\varepsilon^3)$$

From (2-3) we then get the following expression for the first-order force

$$\vec{F}_1 = -K\ \mathrm{Re}\left\{-j\ e^{-j\tau}\sum_{n=1}^{N}\int_{s_{n-1}}^{s_n}(\phi_0+\phi_7)\ \vec{n}ds\right\}$$

ϕ_0 is known from (2-4) and ϕ_7 is known from (3-1). The integrals are computed by the 3-point formula.

When the first-order potential is known we may also compute the mean second-order force. If we take the time average of (1) over one period we obtain

$$\varepsilon^2\ \overline{\vec{F}}^t = -K\int_c \tfrac{1}{2}\overline{(\nabla\hat{\Phi})^2}^t\ \vec{n}ds \tag{4-2}$$

(see f.ex. Ogilvie/4/), where $\overline{A}^t$ means the average of A over one period.

On the cylinder we have

$$(\nabla\hat{\Phi})^2 = \left(\frac{\partial\hat{\Phi}}{\partial s}\right)^2$$

and we therefore get

$$\tfrac{1}{2}\overline{(\nabla\hat{\Phi})^2}^t = \frac{1}{4}\left(\frac{\partial\phi_0}{\partial s}\overline{\frac{\partial\phi_0}{\partial s}} + \frac{\partial\phi_7}{\partial s}\overline{\frac{\partial\phi_7}{\partial s}} + \frac{\partial\phi_0}{\partial s}\overline{\frac{\partial\phi_7}{\partial s}} + \overline{\frac{\partial\phi_0}{\partial s}}\frac{\partial\phi_7}{\partial s}\right)$$

$\frac{\partial\phi_7}{\partial s}$ is computed directly from (3-1) by differentiation of the splines, while $\frac{\partial\phi_0}{\partial s}$ is found from (2-4) and the identity

$$\frac{\partial}{\partial s} = -\sin s \frac{\partial}{\partial x} + \cos s \frac{\partial}{\partial y}$$

This makes it possible to compute the integral in (2) by our standard formula.

5. Results and conclusions

With the method described in this report we have made computations for some different values of the parameters. The phase of the incoming wave, x_0 , has no influence on the results and are assumed to be zero. Therefore we only have to vary the values of K and h (=H/R). We have computed the phase-shift of the transmitted wave and the first- and second-order force components described in the previous section. The results for the phase-shift has been compared with Ogilvie /4/ and Mehlum /2/ and show good agreement. Also the results for the forces can be compared with Ogilvie and show good agreement. This section therefore presents no new results, but they are presented in a somewhat different manner. In tables 1-3 we give some of the numerical values on which the graphs are based.

Fig. 1 shows the delay of the transmitted wave relative to the undisturbed wave. One specific example is shown in fig. 4 where we see that the wave slows down when it approaches and passes over the cylinder, and finally retains it original shape when it emerges on the other side. It is clear from the curves that the phase-shift decays fast when the cylinder is moved away from the free surface.

In fig. 2 we see that the mean force shows a similar behavior as the phase -shift but this time the decrease is even faster when h increases. We also notice that there is a distinct maximum value for $K \approx 0.5$, and this is the same for all values of h.

K \ h	2.0	1.5	1.25	1.125	1.06
0.1	5	6	7	8	10
0.5	26	52	80	113	150
1.0	18	50	91	137	187
1.5	10	38	83	137	200
2.0	4	28	74	133	205
2.5		21	64	127	205
3.0		14	55	120	204
4.0		6	40	105	197
5.0		3	28	90	187
6.0			19	76	174

Table 1 Phase-shift of transmitted wave (in degrees)

K \ h	2.0	1.5	1.25	1.125	1.06
0.1	0.075	0.146	0.293	0.577	1.143
0.5	0.265	0.660	1.535	3.400	7.477
1.0	0.152	0.465	1.088	2.300	4.974
1.5	0.075	0.318	0.827	1.834	4.153
2.0	0.035	0.223	0.662	1.540	3.579
2.5		0.156	0.542	1.320	3.143
3.0		0.109	0.449	1.146	2.805
4.0		0.050	0.311	0.888	2.311
5.0		0.022	0.214	0.702	1.956
6.0			0.145	0.561	1.682

Table 2 Mean vertical force

K \ h	2.0	1.5	1.25	1.125	1.06
0.1	0.575	0.666	0.761	0.860	0.963
0.5	1.106	1.551	1.956	2.264	2.398
1.0	0.701	1.047	1.215	1.155	0.852
1.5	0.375	0.625	0.670	0.481	0.086
2.0	0.183	0.353	0.339	0.122	0.222
2.5		0.191	0.151	0.046	0.294
3.0		0.098	0.051	0.110	0.265
4.0		0.022	0.019	0.122	0.146
5.0		0.003	0.025	0.073	0.057
6.0			0.017	0.040	0.012

Table 3 Amplitude of oscillatory first-order force

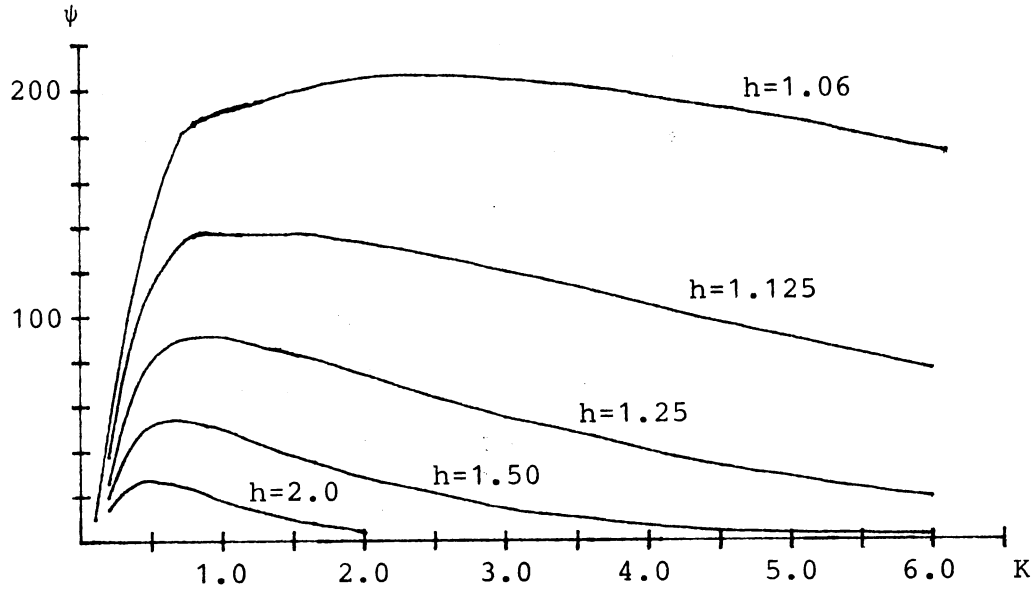


Fig. 5-1 Phase shift of transmitted wave

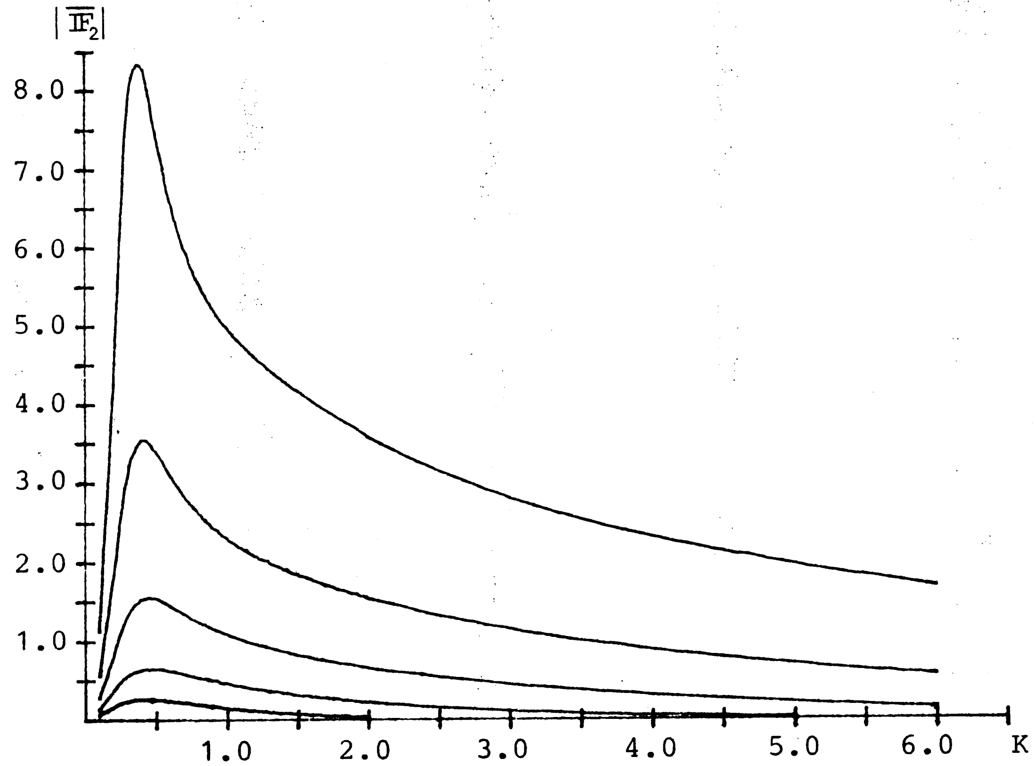


Fig. 5-2 Mean vertical force

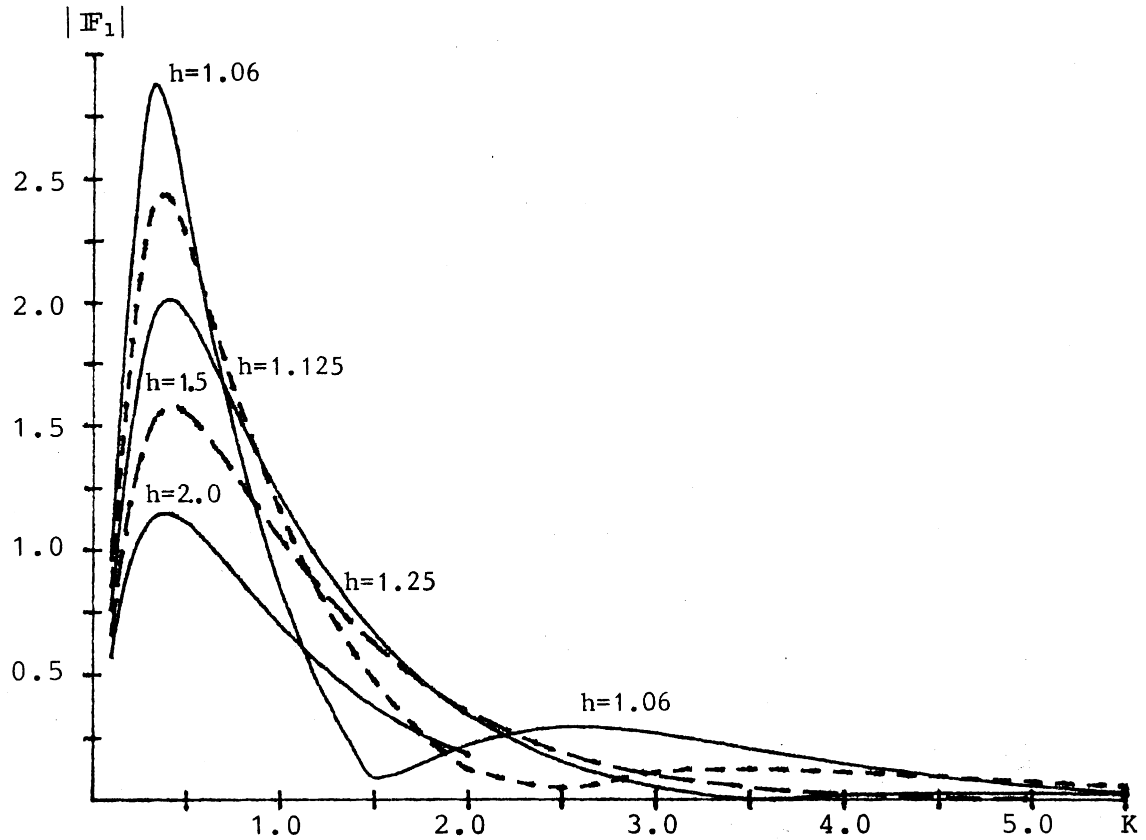


Fig. 5-3 Amplitude of oscillatory force

Also the diagram of the first-order force shows some interesting features. We notice again that there is a distinct maximum value for $K \approx 0.5$ and this is the same for all values of h. This looks like a resonance phenomena, and it probably is. $K = 0.5$ corresponds to a wavelength of 4π, i.e. twice the circumference of the cylinder.

We also see that when the cylinder is very close to the surface the force drops rapidly when the wavelength decreases. In fact there is a wavenumber where the force is almost zero. We therefore have that for wave numbers between 1 and 2 (i.e. wavelengths between π and 2π) the force increases when the cylinder is moved away from the surface, at least as long as h is less than about 1.5. The reason for this is probably that when the cylinder is very close to the surface there is a very strong phase shift.

In these cases it is therefore possible that the incoming potential and the diffraction potential might interfere in such a way that the force is drastically reduced. When the submergence is greater everything is more or less in phase and this can not happen.

The results presented in fig. 3 are quite similar to the results given by Ogilvie /4/ and we therefore expect them to be correct. It must also be mentioned that the horisontal and vertical first-order forces are equal in magnitude, so the results are the same for both components. (See Ogilvie.)

In order to test the method previously described, a set of SIMULA programs was written. Computations were made for a varying number of elements in order to check the convergence of the method. We have been satisfied when the reflection coefficient was less than 0.01.

The most difficult case was as expected $h = 1.06$. If equidistant nodes were used 40 elements were needed in this case to get the desired accuracy, whereas 24 elements were quite sufficient for the other cases. 40 elements give a full 80×80 matrix to solve, but this matrix seems to be very well conditioned because there was no numerical difficulty in solving the system on a 38-bit machine (DEC-10). The total computation time was about 50 CPU-seconds for each wave number on this rather slow machine.

For wave numbers greater than about 1 there is not much happening on the lower half sylinder. A considerable reduction in the computation time was achieved if the number of elements on the lower half cylinder was reduced. It proved sufficient to use 8 elements on this part. If 20 elements were still used on the upper half the computation time in this case was about 25 seconds for each wave number. It is likely that this can be even further reduced by a more careful distribution of the nodes, but we have not investigated this.

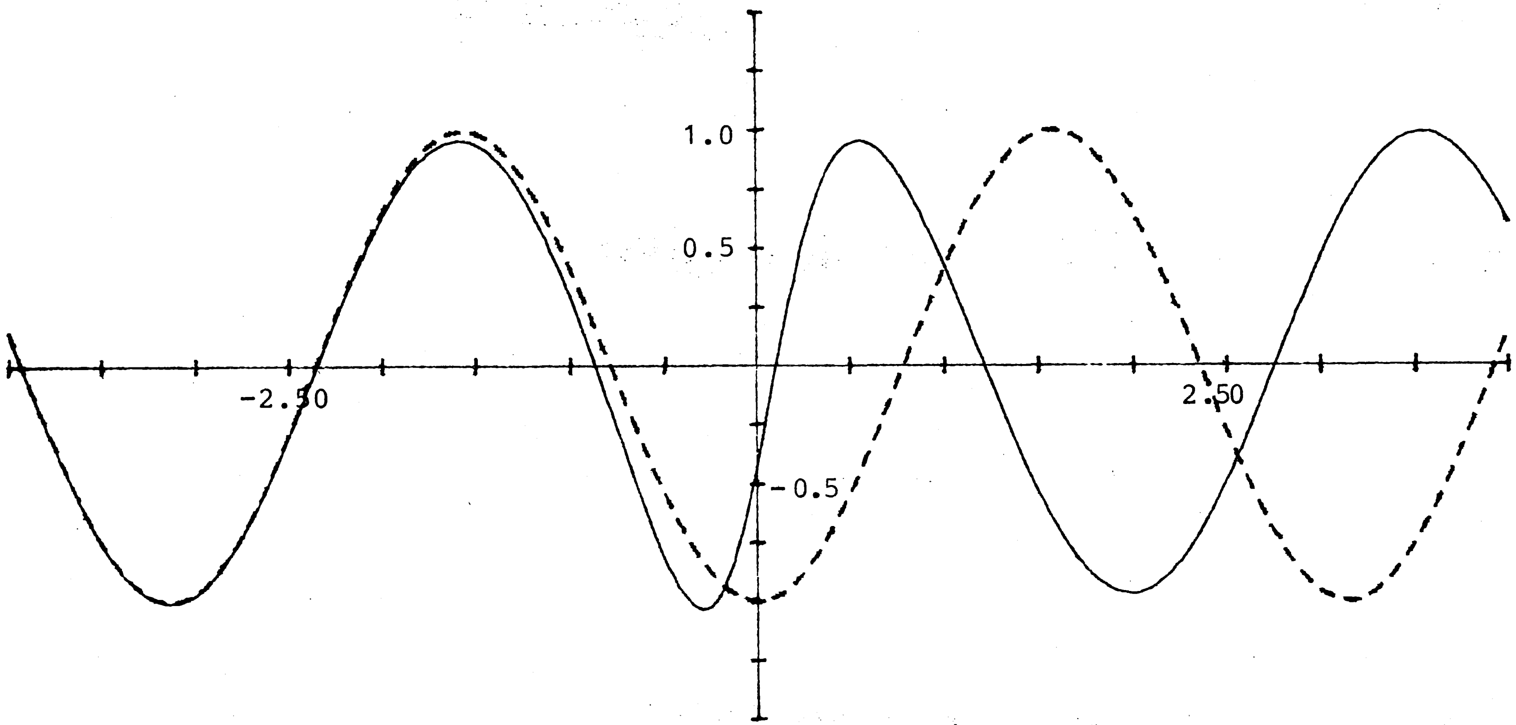


Fig. 5-4 Surface elevation at time $\tau = \frac{\pi}{2} + 2n\pi$, $K = 2.0$, $h = 1.125$
Dotted line: undisturbed wave

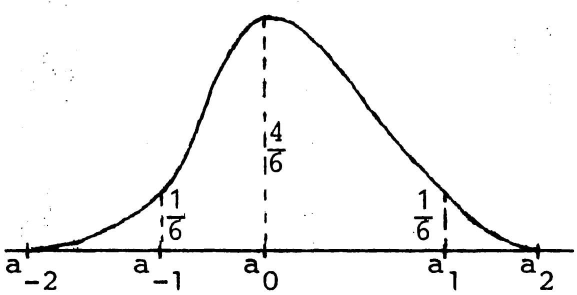
Appendix. Spline approximation

Given 5 points $a_{-2} < a_{-1} < a_0 < a_1 < a_2$ and the function

$$B(x) = \begin{cases} 0 & x < a_{-2} \\ \frac{1}{6}\left(\frac{x-a_{-2}}{a_{-1}-a_{-2}}\right)^3 & a_{-2} < x < a_{-1} \\ \frac{1}{6} + \frac{1}{2}\left(\frac{x-a_{-1}}{a_0-a_{-1}}\right) + \frac{1}{2}\left(\frac{x-a_{-1}}{a_0-a_{-1}}\right)^2 - \frac{1}{2}\left(\frac{x-a_{-1}}{a_0-a_{-1}}\right)^3 & a_{-1} \leq x < a_0 \\ \frac{1}{6} - \frac{1}{2}\left(\frac{x-a_1}{a_1-a_0}\right) + \frac{1}{2}\left(\frac{x-a_1}{a_1-a_0}\right)^2 + \frac{1}{2}\left(\frac{x-a_1}{a_1-a_0}\right)^3 & a_0 \leq x < a_1 \\ -\frac{1}{6}\left(\frac{x-a_2}{a_2-a_1}\right)^3 & a_1 \leq x < a_2 \\ 0 & x > a_2 \end{cases}$$

This is a piecewise cubic polynomial which has continuous first and second derivatives, and it is called the B-spline (or basic (cubic) spline) (see fig. A-1).

Fig. A-1



Let us now study a function $f(x)$ on an interval $[a,b]$. We divide the interval into N subintervals $[x_{i-1}, x_i]$. Let $B_i(x)$ be the B-spline which has its maximum at x_i. These $N+1$ splines form a basis for all splines on these subintervals. The spline approximation to $f(x)$ can therefore be written

$$f(x) \sim \sum_{i=0}^{N} c_i \; B_i(x)$$

where the coefficients c_i are found from some set of equations, depending on the way we have chosen to approximate f. We give two examples:

i) Spline interpolation.

If $f(x_i)$ are given, the interpolating spline is given by the equations

$$\sum_{i=0}^{N} c_i \; B_i(x_k) = f(x_k) \qquad k = 0,1,\ldots,N$$

From the definition of $B(x)$ we see that this is a tridiogonal system.

ii) Spline collocation.

If $f(x)$ is the solution of the equation

$$A \; f(x) = h(x)$$

where A is some operator, we get an approximation to f if we demand that

$$A(\Sigma c_i B_i(x_k)) = h(x_k)$$

i.e. the equation is satisfied at all modal points. If A is the identity operator this of course is the same as i).

If the function $f(x)$ is periodic with period T we may divide one period into N subintervals and write

$$f(x) = \sum_{i=-\infty}^{\infty} c_i \; B_i(x)$$

where $c_{i+N} = c_i$. Therefore all c_i's may be found by solving the N equations we get from one per'od.

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